数理解析研究所講究録
第 788 巻 1992 年 119-127

W-K-B method and uniform asymptotic expansion

by
Toshihiko NISHIMOTO (西本敏彦)
Tokyo Institute of Technology

§ 1 Introduction

In this note, we consider asymptotic properties of solutions of the second order ordinary differential equations of the form

$$(1.1)\qquad \varepsilon^2 \frac{d^2y}{dx^2} = k(x,a)y.$$

when a positive parameter ε tends to zero. The coefficient function $k(x,a)$ is a polynomial of x, which depends on other complex parameters $a=(a_1,a_2,\cdots,a_m)$. More precisely we are mostly concerned with $k(x\ a)$ polynomials of degree two and four,

$$(1.2)\qquad k(x,a) = \pm(x^2 \pm r^2),$$

and

$$(1.3)\qquad k(x,a) = k_0(x-a_1)(x-a_2)(x-a_3)(x-a_4)\ .$$

The zeros of k(x,a) with respect to x are so called turning points of the equation (1.1). These turning points move as the parameters vary and some of turning points may coalesce at certain values of the parameters, which we call critical values of the parameter.

The problems are to construct asymptotic expansions of solutions and of connection formulas between them that are uniformly valid with respect to the parameters in certain regions containing a critical value.

In the previous paper [2], asymptotic expansions of a fundamental system of solutions of the equation (1.1) were constructed in certain regions of x and a. In section 2, the connection formulas of the W-K-B asymptotic solutions of (1.1) with (1.2) are obtained by using those of the parabolic cylinder functions. Let us call these connection formulas as the Weber connection elements for coalescing two turning points, since these formulas must play fundamental role if we want to drive asymptotics expansions of connection formulas for general differential equations (1.1) containing nearly coalescing two turning points. In section 3, the differential equation (1.1) with the coefficient (1.3) is considered as an example, where we assume that a_1 is close to a_2, and a_3 is close to a_4 respectively. This equation appears as a simplified model in the inelastic scattering theory of atomic or molecular collisions [1] . We apply the results of the section 2 in obtaining the connection formulas for critical values of the parameters.

2 Weber connection elements.

In this section, we construct the W-K-B approximations of solutions of the equation (1.1) with (1.2) and their connection formulas.

$$(2.1)\qquad \varepsilon^2 \frac{d^2y}{dx^2} = \pm(x^2 \pm r^2)y\ .$$

These results give us the first terms of the W-K-B approximations and their connection formulas of the solutions of the equations having coalescing two turning points.

There are four cases to consider, which we designate case1,2,3 and 4 with

$$k_1(x,r)=x^2+r^2,$$
$$k_2(x,r)=x^2-r^2,$$
$$k_3(x,r)=r^2-x^2,$$
$$k_4(x,r)=-r^2-x^2.$$

Here the complex parameter r ranges over $0\le r\le R$, and $|\arg r|<\frac{\pi}{2}$ where R is an arbitrarily positive constant.

Case1. $k_1(x,r)=x^2+r^2$.

The solutions of (2.1) are expressed by using the parabolic cylinder function $U(s,b)$ which is defined by the integral

$$\text{(2.2)}\quad U(s,b)=\frac{\Gamma(\frac{1}{2}-b)}{2\pi i}e^{-\frac{1}{4s^2}}\int_C t^{b-\frac{1}{2}}\exp(st-\frac{1}{2}t^2)dt.$$

The function $U(s,b)$ satisfies the differential equation

$$\text{(2.3)}\quad \frac{d^2u}{ds^2}=(\frac{1}{4}s^2+b)u,$$

and has an asymptotic expansion

$$\text{(2.4)}\quad U(s,b)\approx s^{-b-\frac{1}{2}}\{1-\frac{1}{2}(b+\frac{1}{2})(b+\frac{2}{3})s^{-2}+\cdots\}\exp(-\frac{1}{4}s^2)$$

as s tends to infinity in the sector $|\arg s|<\frac{3}{4}\pi.$ It is easy to see that the function $U(s,b)$ is subdominant in the sector $|\arg s|<\frac{1}{4}\pi.$ and dominant in others.

The functions defined by $U(-s,b)$, $U(is,-b)$ and $U(is,-b)$ are also solutions of the equation (2.3). Let M_k be sectors in the complex s-plane such that

$$M_k:-\frac{3}{4}\pi+\frac{\pi}{2}k<\arg s<-\frac{\pi}{4}+\frac{\pi}{2}k.$$

Then the functions $U(-is,-b)$, $U(-s,b)$ and $U(is,-b)$ are subdominant as s tends to infinity in M_2, M_3 and M_4 respectively, and among these four solutions we have the following connection formulas:

$$U(-is,b)=(2\pi)^{-\frac{1}{2}}\Gamma(\frac{1}{2}+b)\{e^{i\pi(\frac{1}{2}b-\frac{1}{4})}U(s,b)+e^{-i\pi(\frac{1}{2}b-\frac{1}{4})}U(-s,b)\},$$

(2.5)

$$U(is,-b)=(2\pi)^{-\frac{1}{2}}\Gamma(\frac{1}{2}+b)\{e^{-i\pi(\frac{1}{2}b-\frac{1}{4})}U(s,b)+e^{i\pi(\frac{1}{2}b-\frac{1}{4})}U(-s,b)\}.$$

Let us define four functions $U(x,r,\varepsilon)$ $(k=1,2,3,4)$ by

$$U_1(x,r,\varepsilon)=U(\sqrt{\frac{2}{\varepsilon}}x,\frac{r^2}{2\varepsilon}),$$

$$U_2(x,r,\varepsilon)=U(-i\sqrt{\frac{2}{\varepsilon}}x,-\frac{r^2}{2\varepsilon}),$$

(2.6)

$$U_3(x,r,\varepsilon)=U(-\sqrt{\frac{2}{\varepsilon}}x,\frac{r^2}{2\varepsilon}),$$

$$U_4(x,r,\varepsilon)=U(i\sqrt{\frac{2}{\varepsilon}}x,-\frac{r^2}{2\varepsilon}).$$

Then these functions are solutions of the equation (2.1). From (2.5) and (2.6), we have

$$U_2(x,r,\varepsilon)=(2\pi)^{-\frac{1}{2}}\Gamma(\frac{1}{2}+\frac{r^2}{2\varepsilon})\{e^{\frac{i\pi}{4}(\frac{r^2}{\varepsilon}-1)}U_1(x,r,\varepsilon)+e^{-\frac{i\pi}{4}(\frac{r^2}{\varepsilon}-1)}U_3(x,r,\varepsilon)\},$$

(2.7)

$$U_4(x,r,\varepsilon)=(2\pi)^{-\frac{1}{2}}\Gamma(\frac{1}{2}+\frac{r^2}{2\varepsilon})\{e^{-\frac{i\pi}{4}(\frac{r^2}{\varepsilon}-1)}U_1(x,r,\varepsilon)+e^{\frac{i\pi}{4}(\frac{r^2}{\varepsilon}-1)}U_3(x,r,\varepsilon)\}.$$

From the asymptotic formulas (2.4) of $U(s,b)$, we have asymptotic expansions of the functions $U_k(x,r,\varepsilon)$ as $\sqrt{2/\varepsilon x}$ goes to infinity:

$$U_1(x,r,\varepsilon)\approx\left(\sqrt{\frac{2}{\varepsilon}}x\right)^{-\frac{r^2}{2\varepsilon}-\frac{1}{2}}\exp\left(-\frac{x^2}{2\varepsilon}\right), \qquad |\arg x|<\frac{3}{4}\pi,$$

$$U_2(x,r,\varepsilon)\approx\left(-i\sqrt{\frac{2}{\varepsilon}}x\right)^{\frac{r^2}{2\varepsilon}-\frac{1}{2}}\exp\left(\frac{x^2}{2\varepsilon}\right), \qquad |\arg(-ix)|<\frac{3}{4}\pi,$$

(2.8)

$$U_3(x,r,\varepsilon)\approx\left(-\sqrt{\frac{2}{\varepsilon}}x\right)^{-\frac{r^2}{2\varepsilon}-\frac{1}{2}}\exp\left(-\frac{x^2}{2\varepsilon}\right), \qquad |\arg(-x)|<\frac{3}{4}\pi,$$

$$U_4(x,r,\varepsilon)\approx\left(i\sqrt{\frac{2}{\varepsilon}}x\right)^{\frac{r^2}{2\varepsilon}-\frac{1}{2}}\exp\left(\frac{x^2}{2\varepsilon}\right), \qquad |\arg(ix)|<\frac{3}{4}\pi.$$

On the other hand, the equation (2.1) has solutions whose asymptotic expansions are of the W-K-B types as follows:

$$
(2.9)\quad
\begin{aligned}
V_1(x,r,\varepsilon) &\approx \left(x^2+r^2\right)^{-\frac{1}{4}} \exp\{-\frac{1}{\varepsilon}\int_{ri}^{x}\left(x^2+r^2\right)^{\frac{1}{2}}dx\},\\
V_2(x,r,\varepsilon) &\approx \left(x^2+r^2\right)^{-\frac{1}{4}} \exp\{\frac{1}{\varepsilon}\int_{ri}^{x}\left(x^2+r^2\right)^{\frac{1}{2}}dx\},\\
V_3(x,r,\varepsilon) &\approx \left(x^2+r^2\right)^{-\frac{1}{4}} \exp\{-\frac{1}{\varepsilon}\int_{-ri}^{x}\left(x^2+r^2\right)^{\frac{1}{2}}dx\},\\
V_4(x,r,\varepsilon) &\approx \left(x^2+r^2\right)^{-\frac{1}{4}} \exp\{\frac{1}{\varepsilon}\int_{-ri}^{x}(x^2+r^2)^{\frac{1}{2}}dx\},
\end{aligned}
$$

The solutions $V_i(x,r,\varepsilon)$ is subdominant as $\varepsilon \to 0$ for x in M_i of the complex x-plane, and at the same time as x tends to infinity in M_i for fixed ε positive. We call these solutions $V_i(x,r,\varepsilon)$ as W-K-B type solutions of the equations (2.1).

Since both functions $U_i(x,r,\varepsilon)$ and $V_i(x,r,\varepsilon)$ are solutions of the equation (2.1) and subdominant as x tends to infinity in M_i , we must have

(2.10) $\qquad U_i(x,r,\varepsilon)=h_i(r,\varepsilon)V_i(x,r,\varepsilon) \quad (i=1,2,3,4).$

Here the coefficient $h_i(r,\varepsilon)$ can be obtained from

$$h_i(r,\varepsilon)=\lim_{x\to\infty,x\in M_i}\frac{U_i(x,r,\varepsilon)}{V_i(x,r,\varepsilon)}.$$

From the asymptotic formula (2.4) of $U(s,b)$, we have

$$(2.11)\qquad U_1(x,r,\varepsilon)\approx\left(\sqrt{\frac{2}{\varepsilon}}\right)^{-\frac{r^2}{2\varepsilon}-\frac{1}{2}}\exp\left(-\frac{x^2}{2\varepsilon}\right),\quad |\arg x|<\frac{3}{4}\pi.$$

And we have as x tends to infinity

$$(2.12)\qquad V_1(x,r,\varepsilon)\approx x^{-\frac{1}{2}}\exp\{-\frac{1}{\varepsilon}(\frac{1}{2}x^2+\frac{r^2}{4}+\frac{r^2}{2}\log\frac{2x}{ri})\}.$$

Then from (2.11) and (2.12) we get

$$(2.13)\qquad h_1(r,\varepsilon)=\left(\frac{2}{\varepsilon}\right)^{-\frac{1}{4}\left(\frac{r^2}{\varepsilon}+1\right)}\left(\frac{2}{ri}\right)^{\frac{r^2}{2\varepsilon}}\exp\left(\frac{r^2}{4\varepsilon}\right).$$

By the same way, other coefficients can be obtained as follows:

$$h_2(r,\varepsilon)=\left(-i\sqrt{\frac{2}{\varepsilon}}\right)^{\frac{r^2}{2\varepsilon}-\frac{1}{2}}\left(\frac{2}{ri}\right)^{-\frac{r^2}{2\varepsilon}}\exp\left(-\frac{r^2}{4\varepsilon}\right),$$

$$h_3(r,\varepsilon)=\left(-\sqrt{\frac{2}{\varepsilon}}\right)^{-\left(\frac{r^2}{2\varepsilon}+\frac{1}{2}\right)}\left(-\frac{2}{ri}\right)^{\frac{r^2}{2\varepsilon}}\exp\left(\frac{r^2}{4\varepsilon}\right),$$

$$h_4(r,\varepsilon)=\left(i\sqrt{\frac{2}{\varepsilon}}\right)^{\frac{r^2}{2\varepsilon}-\frac{1}{2}}\left(-\frac{2}{ri}\right)^{-\frac{r^2}{2\varepsilon}}\exp\left(-\frac{r^2}{4\varepsilon}\right).$$

From (2.7) and (2.9) we can obtain the connection formulas between the W-K-B type solutions. Let the connection formulas be

$$V_2 = k_{21}V_1 + k_{23}V_3,$$

(2.14)

$$V_4 = k_{41}V_1 + k_{43}V_3,$$

then

$$k_{21} = (2\pi)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}+\frac{r^2}{2\varepsilon}\right)e^{\frac{i\pi}{4}\left(\frac{r^2}{\varepsilon}-1\right)}h_2(r,\varepsilon)^{-1}h_1(r,\varepsilon),$$

$$k_{23} = (2\pi)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}+\frac{r^2}{2\varepsilon}\right)e^{-\frac{i\pi}{4}\left(\frac{r^2}{\varepsilon}-1\right)}h_2(r,\varepsilon)^{-1}h_3(r,\varepsilon)$$

(2.15)

$$k_{41} = (2\pi)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}+\frac{r^2}{2\varepsilon}\right)e^{-\frac{i\pi}{4}\left(\frac{r^2}{\varepsilon}-1\right)}h_4(r,\varepsilon)^{-1}h_1(r,\varepsilon),$$

$$k_{43} = (2\pi)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}+\frac{r^2}{2\varepsilon}\right)e^{\frac{i\pi}{4}\left(\frac{r^2}{\varepsilon}-1\right)}h_4(r,\varepsilon)^{-1}h_3(r,\varepsilon).$$

Case 2. $k_2(x,r) = x^2 - r^2$

In this case the definition of $U_i(x,r,\varepsilon)$ and $V_i(x,r,\varepsilon)$ $(i = 1,2,3,4)$ are given by (2.6) and (2.9) respectively with $-ir$ in place of r. Accordingly the connection formulas (2.7), asymptotic forms of the coefficients (2.13) and (2.15) are valid by replacing r by $-ir$.

Case 3. $k_3(x,r) = -x^2 + r^2$.

In this case, we substitute $se^{i\pi/4}$ and $ae^{-i\pi/4}$ for x and r, and we get the equation of the case 1. Thus the results for the case 1 can be applied, in particular the asymptotic form (2.15) of the connection formulas are valid with $re^{i\pi/4}$ in place of r.

Case 4. $k_4(x,r) = -x^2 - r^2$.

This case can also be transformed into the case 1 by replacing $se^{i\pi/4}$ and $ae^{i\pi/4}$ for x and r, so that the connection formulas (2.15) is valid with $re^{-i\pi/4}$ instead of r.

3. Application

In this section, let us consider a differential equation

(3.1) $$\varepsilon^2\frac{d^2y}{dx^2} = k(x,a)y$$

with $k(x,a) = k(x-a_1)(x-a_2)(x-a_3)(x-a_4)$,

where k is constant. We assume that the Stokes curve configuration for (3.1) is as in Fig.1, turning points a_1 and a_2 are close with each other and the same for two turning points $a_3$and a_4 , but two pairs of coalescing two turning points are well seperated.

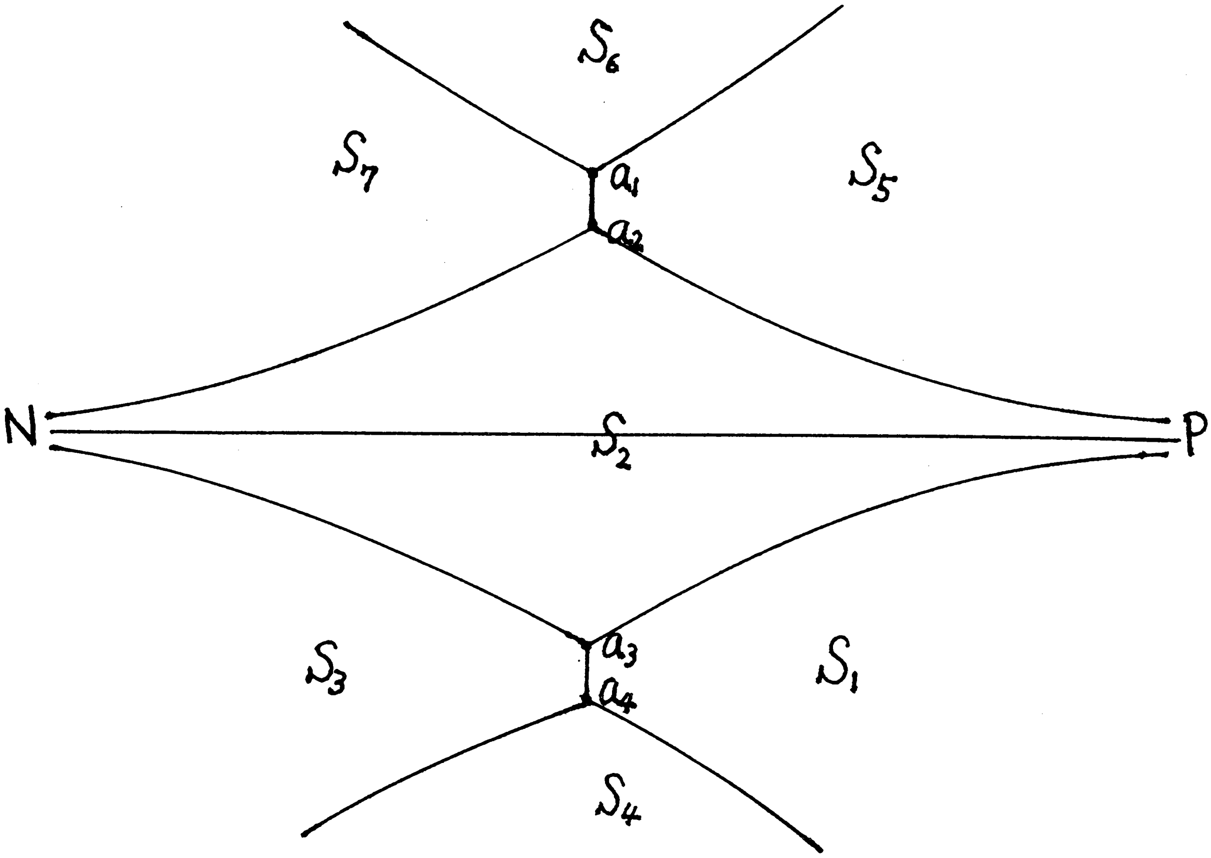

Fig.1

By the Liouville transformation, the equation (3.1) is approximately changed into the equation (2.1), and so the results of the section 2 are applied.

(1) The lower Liouville transformation from x,y to s,u is defined by

$$(3.2)\qquad x=x(s),\qquad u(s)=\left(\frac{dx}{ds}\right)^{-\frac{1}{2}}y(x,s),$$

by which the equation (3.1) becomes

$$(3.3)\quad \varepsilon^2\frac{d^2u}{ds^2}=\{\left(\frac{ds}{dx}\right)^2k(x,a)+\varepsilon^2\left(\frac{dx}{ds}\right)^{\frac{1}{2}}\frac{d^2}{ds^2}\left(\left[\frac{ds}{dx}\right]^{-\frac{1}{2}}\right)\}u.$$

Here the transformation $x=x(s)$ is chosen such that

$$\left(\frac{dx}{ds}\right)^2k(x,a)=s^2+r^2,$$

$$\int_{a_3}^{x}\{k(x,a)\}^{\frac{1}{2}}dx=\int_{ri}^{s}(s^2+r^2)^{\frac{1}{2}}ds,$$

$$\int_{a_3}^{a_4}\{k(x,a)\}^{\frac{1}{2}}dx=\int_{ri}^{-ri}(s^2+r^2)^{\frac{1}{2}}ds.$$

Then the equation (3.1) becomes

(3.4) $$\varepsilon^2 \frac{d^2u}{ds^2} = \{s^2 + r^2 + O(\varepsilon^2)\}u,$$
and from results of the previous paper [2] ,we can construct the W-K-B type approximations of solutions of (3.1) as

$$y_i(x,r,\varepsilon) = \left(\frac{dx}{ds}\right)^{\frac{1}{2}} V_i(s,r,\varepsilon), \quad (i = 1,2,3,4),$$

where V_i are defined in (2.9) with s in place of x.
Thus we have

$$y_1(x,a,\varepsilon) = k(x,a,\varepsilon)^{-\frac{1}{4}} \exp[-\frac{1}{\varepsilon}\int_{a_3}^{x} k(x,a,\varepsilon)^{\frac{1}{2}} dx], \quad subd.inS_1,$$

$$y_2(x,a,\varepsilon) = k(x,a,\varepsilon)^{-\frac{1}{4}} \exp[\frac{1}{\varepsilon}\int_{a_3}^{x} k(x,a,\ \varepsilon)^{\frac{1}{2}} dx], \quad subd.inS_2 \cup S_5,$$

(3.5)

$$y_3(x,a,\varepsilon) = k(x,a,\varepsilon)^{-\frac{1}{4}} \exp[-\frac{1}{\varepsilon}\int_{a_4}^{x} k(x,a,\varepsilon)^{\frac{1}{2}} dx], \quad subd.inS_3,$$

$$y_4(x,a,\varepsilon) = k(x,a,\varepsilon)^{-\frac{1}{4}} \exp[\frac{1}{\varepsilon}\int_{a_4}^{x} k(x,a,\varepsilon)^{\frac{1}{2}} dx], \quad subd.inS_4,$$

where subd in S means that the function is exponentially subdominant in the region S.

The reading terms of the connection formulas

(3.6) $$y_2(x,a,\varepsilon) = K_{21}y_1(x,a,\varepsilon) + K_{23}y_3(x,a,\varepsilon)$$
are obtained from the formulas (2.15)

$$K_{21} \approx k_{21}(r,\varepsilon), \qquad K_{23} \approx k_{23}(r,\varepsilon),$$

with

$$r = \left[\frac{2i}{\pi}\int_{a_3}^{a_4} k(x,a,\varepsilon)^{\frac{1}{2}} dx\right]^{\frac{1}{2}}, \qquad (\operatorname{Im} ri > 0).$$

(2) Analogously we make the upper Liouville transformation (3.2) to the equation (3.1) with

$$\int_{a_1}^{x} k(x,a,\varepsilon)^{\frac{1}{2}} dx = \int_{\tilde{r}i}^{s} (s^2 + \tilde{r}^2)^{\frac{1}{2}} ds,$$

$$\int_{a_1}^{a_2} k(x,a,\varepsilon)^{\frac{1}{2}} dx = \int_{\tilde{r}i}^{-\tilde{r}i} (s^2 + \tilde{r}^2)^{\frac{1}{2}} ds = -\frac{\pi i \tilde{r}^2}{2},$$

and get

$$\varepsilon^2 \frac{d^2u}{ds^2} = \{(s^2 + \tilde{r}^2) + O(\varepsilon^2)\}u.$$

Therefore we obtain another set of W-K-B type approximations of solutions of (3.1) as follow:

$$z_1(x,a,\varepsilon)=k(x,a,\varepsilon)^{-\frac{1}{4}}\exp[-\frac{1}{\varepsilon}\int_a^x k(x,a,\varepsilon)^{\frac{1}{2}}dx],\ subd.inS_5,$$

$$z_2(x,a,\varepsilon)=k(x,a,\varepsilon)^{-\frac{1}{4}}\exp[\frac{1}{\varepsilon}\int_{a_1}^x k(x,a,\ \varepsilon)^{\frac{1}{2}}dx],\ \ subd.inS_6,$$

(3.7)

$$z_3(x,a,\varepsilon)=k(x,a,\varepsilon)^{-\frac{1}{4}}\exp[-\frac{1}{\varepsilon}\int_{a_2}^x k(x,a,\varepsilon)^{\frac{1}{2}}dx],\ subd.inS_7,$$

$$z_4(x,a,\varepsilon)=k(x,a,\varepsilon)^{-\frac{1}{4}}\exp[\frac{1}{\varepsilon}\int_{a_2}^x k(x,a,\varepsilon)^{\frac{1}{2}}dx],\ \ subd.inS_1\cup S_2,$$

and we have

(3.8) $$z_4(x,a,\varepsilon)=\tilde{K}_{41}z_1(x,a,\varepsilon)+\tilde{K}_{43}z_3(x,a,\varepsilon)$$

with

$$\tilde{K}_{41}\approx k_{41}(\tilde{r},\varepsilon),\qquad \tilde{K}_{43}\approx k_{43}(\tilde{r},\varepsilon),$$

where

$$\tilde{r}=\left[\frac{2i}{\pi}\int_{a_1}^{a_2}k(x,a,\varepsilon)^{\frac{1}{2}}dx\right]^{\frac{1}{2}},\qquad (\mathrm{Im}\,\tilde{r}i>0).$$

Lastly we calculate the asymptotic forms of connection formulas between oscillatoory solutions as x tends to infinity. Note th at the following table of statements is correct:

solution	*subd.in*	*wave*
y_1	S_1	ingoing wave on a_3P,
y_2	$S_2\cup S_5$	outgoing wave on a_3P,
y_3	S_3	ingoing wave on a_3N,
z_1	S_5	*outgoing wave on* a_2P,
z_3	S_7	*outgoing wave on* a_2N
z_4	$S_1\cup S_2$	*ingoing wave on* a_2P.

Since $y_2(x,a,\varepsilon)$ and $z_1(x,a,\varepsilon)$ are both subdominant as $x\to\infty$ in S_5, it must be

$$y_2(x,a,\varepsilon)=\left[\exp\frac{1}{\varepsilon}\int_{a_3}^{a_1}k(x,a,\varepsilon)^{\frac{1}{2}}dx\right]z_1(x,a,\varepsilon), \tag{3.9}$$

$$\mathrm{Re}\int_{a_3}^{a_1}k(x,a,\varepsilon)^{\frac{1}{2}}dx<0,$$

and we have

$$z_4(x,a,\varepsilon)=\left[\exp\frac{1}{\varepsilon}\int_{a_2}^{a_3}k(x,a,\varepsilon)^{\frac{1}{2}}dx\right]y_1(x,a,\varepsilon). \tag{3.10}$$

The solutions y_1 and z_1 are ingoing wave and outgoing wave as $x\to\infty$, and y_3 and z_3 are ingoing and outgoing wave as $x\to-\infty$, respectively. Then we have from (3.6), (3.8), (3.9) and (3.10) the following connection formula between (y_3,z_3) and (y_1,z_1):

$$y_3(x,a,\varepsilon)=Ay_1(x,a,\varepsilon)+Bz_1(x,a,\varepsilon),$$
$$z_3(x,a,\varepsilon)=Cy_1(x,a,\varepsilon)+Dz_1(x,a,\varepsilon),$$

with

$$A\approx-k_{23}(r,\varepsilon)^{-1}k_{21}(r,\varepsilon),$$

$$B\approx k_{23}(r,\varepsilon)\exp\frac{1}{\varepsilon}\int_{a_3}^{a_1}k(x,a,\varepsilon)^{\frac{1}{2}}dx,$$

$$C\approx k_{43}(\tilde{r},\varepsilon)\exp\frac{1}{\varepsilon}\int_{a_2}^{a_3}k(x,a,\varepsilon)^{\frac{1}{2}}dx,$$

$$D\approx-k_{43}(\tilde{r},\varepsilon)^{-1}k_{41}(\tilde{r},\varepsilon).$$

References

[1] Nakamura,H., Dynamical repre. and nonadiabatic electronic transition in atomic collisions, Physical Review A 26(1982) 3125- 3136.

[2] Nishimoto,T., Confluent W-K-B approximations,1, Kodai Math. J., 10(1987) 362-374.